THE STRAIN OF A NON-GRAVITATING SPHERE OF VARIABLE DENSITY[*]

BY

L. M. HOSKINS

§ 1. *Introductory.*

The mathematical problem of the strain of an elastic sphere is of special physical interest because of its relation to the question of the elastic yielding of the earth to disturbing forces. The cases of the problem which have been solved, however, involve assumptions which depart from what is known or thought probable in the case of the earth. The solutions of LAMÉ and KELVIN apply to a homogeneous body acted upon by bodily forces which are known functions of the coördinates. This case disagrees with the facts as regards the earth in two particulars: (1) The earth is certainly far from homogeneous in density and is very likely far from homogeneous in elastic properties. (2) The important bodily forces of self-gravitation cannot be expressed as functions of the coördinates in advance of the solution, since the changes in these forces caused by the strain are of the same order of importance as the external disturbing forces; it is only by assuming incompressibility that it becomes possible to express the gravitational forces without the use of unknown functions.

In a previous paper[†] was given a solution of the problem for a compressible sphere initially of uniform density, taking account of the changes in the gravitational forces caused by the strain.

In the present paper will be found a solution for the case of a non-gravitating sphere of which the density is a given function of the distance from the center, the elastic moduli being assumed to be uniform in value throughout the body.

By combining these two cases, and introducing the additional assumption that the elastic moduli are functions of the distance from the center, we should have perhaps the nearest possible approximation to a completely general statement of the problem presented by the actual earth. While there is no difficulty in forming the differential equations for this general problem in a manner analogous to that employed in the two cases which have been solved, a solution of these general equations has not thus far been obtained.

The case treated in this paper appears, however, to possess some interest and importance, being a generalization in one important particular of the problem which formed the basis of the original estimates of the rigidity of the earth.

§ 2. *Statement of Problem.*

The problem here treated is the following:

To determine the strain of an elastic sphere of which the density is a given function of the distance from the center, under the action of bodily forces having a potential, but free from stress on the bounding surface. The potential may be any function which, when expressed in polar coördinates, is developable as to θ and ϕ in a series of spherical surface harmonics.

§ 3. *Equations of Equilibrium of Volume-Element.*

Using polar coördinates r, θ, ϕ, let u_r, u_θ, u_ϕ be the elastic displacements, ϖ_r, ϖ_θ, ϖ_ϕ the rotation-components, Δ the cubical dilatation, F_r, F_θ, F_ϕ the components of bodily force per unit mass, $\lambda + \tfrac{2}{3}\mu$, μ the bulk-modulus and rigidity-modulus, ρ the density.

The three equations of equilibrium for an element of volume are the following:

(1) $\qquad (\lambda + 2\mu)\dfrac{\partial \Delta}{\partial r} - \dfrac{2\mu}{r \sin \theta}\left[\dfrac{\partial}{\partial \theta}(\varpi_\phi \sin \theta) - \dfrac{\partial \varpi_\phi}{\partial \phi}\right] + \rho F_r = 0,$

(2) $\qquad (\lambda + 2\mu)\dfrac{1}{r}\dfrac{\partial \Delta}{\partial \theta} - \dfrac{2\mu}{r \sin \theta}\left[\dfrac{\partial \varpi_r}{\partial \phi} - \dfrac{\partial}{\partial r}(r\varpi_\phi \sin \theta)\right] + \rho F_\theta = 0,$

(3) $\qquad (\lambda + 2\mu)\dfrac{1}{r \sin \theta}\dfrac{\partial \Delta}{\partial \phi} - \dfrac{2\mu}{r}\left[\dfrac{\partial}{\partial r}(r\varpi_\theta) - \dfrac{\partial \varpi_r}{\partial \theta}\right] + \rho F_\phi = 0;$

in which

(4) $\qquad \Delta = \dfrac{1}{r^2 \sin \theta}\left[\dfrac{\partial}{\partial r}(r^2 u_r \sin \theta) + \dfrac{\partial}{\partial \theta}(r u_\theta \sin \theta) + \dfrac{\partial}{\partial \phi}(r u_\phi)\right],$

(5) $\qquad 2\varpi_r = \dfrac{1}{r^2 \sin \theta}\left[\dfrac{\partial}{\partial \theta}(r u_\phi \sin \theta) - \dfrac{\partial}{\partial \phi}(r u_\theta)\right],$

(6) $\qquad 2\varpi_\theta = \dfrac{1}{r \sin \theta}\left[\dfrac{\partial u_r}{\partial \phi} - \dfrac{\partial}{\partial r}(r u_\phi \sin \theta)\right],$

(7) $\qquad 2\varpi_\phi = \dfrac{1}{r}\left[\dfrac{\partial}{\partial r}(r u_\theta) - \dfrac{\partial u_r}{\partial \theta}\right].$

Let W denote the potential of the bodily forces, and assume that

(8) $\qquad\qquad\qquad\qquad W = \Sigma R_i S_i,$

in which S_i denotes a spherical surface harmonic of order i, and R_i a function of r. Then

$$F_r = \Sigma \frac{dR_i}{dr} S_i,$$

(9)
$$F_\theta = \Sigma \frac{R_i}{r} \frac{\partial S_i}{\partial \theta},$$

$$F_\phi = \Sigma \frac{R_i}{r \sin \theta} \frac{\partial S_i}{\partial \phi}.$$

A solution of equations (1), (2) and (3) may be formed by putting together solutions corresponding to the separate terms in the value of W. We therefore confine the following discussion to the solution corresponding to a single term $R_i S_i$.

We now assume

(10) $$u_r = u S_i, \qquad u_\theta = v \frac{\partial S_i}{\partial \theta}, \qquad u_\phi = \frac{v}{\sin \theta} \frac{\partial S_i}{\partial \phi},$$

in which u and v are functions of r only. Substituting these values in equations (4), $\cdots$, (7), and making use of the differential equation

(11) $$\frac{1}{\sin^2 \theta} \frac{\partial^2 S_i}{\partial \phi^2} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial S_i}{\partial \theta} \right) + i(i+1) S_i = 0$$

which is satisfied by any spherical surface harmonic of order i, we find

(12) $$\Delta = \left[\frac{1}{r^2} \frac{d(r^2 u)}{dr} - i(i+1) \frac{v}{r} \right] S_i,$$

(13) $$2\varpi_r = 0,$$

(14) $$2\varpi_\theta = \frac{1}{r} \left[u - \frac{d(rv)}{dr} \right] \frac{1}{\sin \theta} \frac{\partial S_i}{\partial \phi},$$

(15) $$2\varpi_\phi = -\frac{1}{r} \left[u - \frac{d(rv)}{dr} \right] \frac{\partial S_i}{\partial \theta}.$$

If the foregoing values of F_r, F_θ, F_ϕ, Δ, ϖ_r, ϖ_θ and ϖ_ϕ be substituted in (1), (2) and (3), these are freed from θ and ϕ. Equations (2) and (3) in fact become identical, and there remain to be satisfied two ordinary differential equations for determining u, v as functions of r. Writing R for R_i, these equations are the following:

(16)
$$(\lambda + 2\mu) \frac{d}{dr} \left(\frac{1}{r^2} \frac{d(r^2 u)}{dr} - i(i+1) \frac{v}{r} \right)$$
$$+ \frac{i(i+1)\mu}{r^2} \left(\frac{d(rv)}{dr} - \right) + \rho \frac{dR}{dr} = 0,$$

$$(17) \quad (\lambda + 2\mu)\left(\frac{1}{r^2}\frac{d(r^2 u)}{dr} - i(i+1)\frac{v}{r}\right) + \mu\frac{d}{dr}\left(\frac{d(rv)}{dr} - u\right) + \rho R \doteq 0.$$

It is now convenient to replace r by $x = r/a$ as independent variable, and to introduce y and z, defined by the equations

$$(18) \qquad y = \frac{\Delta}{S_i} = \frac{1}{r^2}\frac{d(r^2 u)}{dr} - i(i+1)\frac{v}{r},$$

$$(19) \qquad z = \frac{2\varpi_\phi}{\partial S_i/\partial\theta} = \frac{1}{r}\left(\frac{d(rv)}{dr} - u\right),$$

thus reducing the differential equations to the following:

$$(20) \qquad (\lambda + 2\mu)\frac{dy}{dx} + i(i+1)\mu\frac{z}{x} + \rho\frac{dR}{dx} = 0,$$

$$(21) \qquad (\lambda + 2\mu)y + \mu\frac{d(xz)}{dx} + \rho R = 0.$$

§4. *Surface Conditions.*

We now have to express the condition that the bounding surface is free from stress, that is,

$$(22) \qquad \widehat{rr} = 0, \qquad \widehat{r\theta} = 0, \qquad \widehat{r\phi} = 0 \qquad\qquad (x=1).$$

Using (10) in the general formulas for these stresses, we find

$$(23) \qquad \widehat{rr} = \lambda\Delta + 2\mu\frac{\partial u_r}{\partial r} = \left(\lambda y + 2\mu\frac{du}{dr}\right)S_i,$$

$$(24) \qquad \widehat{r\theta} = \mu\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right) = \mu\left(\frac{dv}{dr} - \frac{v}{r} + \frac{u}{r}\right)\frac{\partial S_i}{\partial\theta},$$

$$(25) \quad \widehat{r\phi} = \mu\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}\right) = \mu\left(\frac{dv}{dr} - \frac{v}{r} + \frac{u}{r}\right)\frac{1}{\sin\theta}\frac{\partial S_i}{\partial\phi},$$

so that equations (22) reduce to the two

$$(26) \qquad \lambda y + 2\mu\frac{du}{dr} = 0,$$

$$(27) \qquad \frac{dv}{dr} - \frac{v}{r} + \frac{u}{r} = 0,$$

to be satisfied when $x = 1$.

§ 5. *Solution of Differential Equations.*

By eliminating successively z and y between equations (20) and (21) we obtain the two equations

$$(28) \quad \left(x^2 \frac{d^2}{dx^2} + 2x \frac{d}{dx} - i(i+1)\right)\left((\lambda + 2\mu)y + \rho R\right) - \frac{d}{dx}\left(x^2 R \frac{d\rho}{dx}\right) = 0,$$

$$(29) \quad \mu\left(x^2 \frac{d^2 z}{dx^2} + 2x \frac{dz}{dx} - i(i+1)z\right) + x R \frac{d\rho}{dx} = 0.$$

The solutions of these equations are the following:

$$(30) \quad (2i+1)[(\lambda + 2\mu)y + \rho R] = x^i \int x^{-i-1} \frac{d}{dx}\left(x^2 R \frac{d\rho}{dx}\right) dx$$
$$- x^{-i-1} \int x^i \frac{d}{dx}\left(x^2 R \frac{d\rho}{dx}\right) dx,$$

$$(31) \quad (2i+1)\mu z = - x^i \int x^{-i-1}\left(x R \frac{d\rho}{dx}\right) dx + x^{-i-1} \int x^i \left(x R \frac{d\rho}{dx}\right) dx,$$

in which, for the general solutions, each integral must be understood to involve an arbitrary constant. These constants are here omitted, since they are accounted for in the complementary solutions to be obtained presently. Modifying by integration by parts, we find

$$(32) \quad (\lambda + 2\mu)y = \frac{1}{2i+1}\left[(i+1)x^i \int \rho \frac{d}{dx}\left(\frac{R}{x^i}\right) dx + \frac{i}{x^{i+1}} \int \rho \frac{d(Rx^{i+1})}{dx} dx\right],$$

$$(33) \quad \mu z = \frac{1}{2i+1}\left[x^i \int \rho \frac{d}{dx}\left(\frac{R}{x^i}\right) dx - \frac{1}{x^{i+1}} \int \rho \frac{d(Rx^{i+1})}{dx} dx\right].$$

These values of y and z may now be substituted in (18) and (19), giving two simultaneous linear differential equations for determining u and v. The solutions of these equations may conveniently be expressed as follows:

From (18) and (19) we readily obtain the equations

$$(34) \quad \frac{d}{dx}(iv - u) + \frac{i+2}{x}(iv - u) = a(iz - y),$$

$$(35) \quad \frac{d}{dx}[(i+1)v + u] - \frac{i-1}{x}[(i+1)v + u] = a[(i+1)z + y],$$

of which the solutions are

$$(36) \quad iv - u = \frac{a}{x^{i+2}} \int (iz - y) x^{i+2} dx,$$

(37) $$(i+1)v + u = ax^{i-1}\int \frac{(i+1)z + y}{x^{i-1}}\, dx.$$

These are easily solved for u and v.

The forms of the complementary solutions are easily seen by considering the constants of integration in (32), (33), (36) and (37). The values of the constants must be determined so as to satisfy (18), (19), (20) and (21). The complementary solutions are thus found to be the following:

$$y = C_i x^i + C_{-i-1} x^{-i-1},$$

$$z = D_i x^i + D_{-i-1} x^{-i-1},$$

(38)

$$\frac{u}{r} = A_i x^i + A_{-i-1} x^{-i-1} + A_{i-2} x^{i-2} + A_{-i-3} x^{-i-3},$$

$$\frac{v}{r} = B_i x^i + B_{-i-1} x^{-i-1} + B_{i-2} x^{i-2} + B_{-i-3} x^{-i-3};$$

in which the twelve constants must satisfy the eight equations

$$(\lambda + 2\mu)C_i + (i+1)\mu D_i = 0,$$

$$(i+3)A_i - i(i+1)B_i - C_i = 0,$$

$$A_i - (i+2)B_i + D_i = 0,$$

$$(\lambda + 2\mu)C_{-i-1} - i\mu D_{-i-1} = 0,$$

(39)

$$(i-2)A_{-i-1} + i(i+1)B_{-i-1} + C_{-i-1} = 0,$$

$$A_{-i-1} + (i-1)B_{-i-1} + D_{-i-1} = 0,$$

$$A_{i-2} - iB_{i-2} = 0,$$

$$A_{-i-3} + (i+1)B_{-i-3} = 0.$$

The complete solutions will thus involve four arbitrary constants; by assigning values to these we may satisfy four equations of condition at the boundary.

This solution covers the case of a spherical shell with bounding surfaces free from stress, or with arbitrary surface stress of a restricted type (corresponding to the restriction imposed on the potential of the bodily forces), or with arbitrary surface displacements of a similarly restricted type.

For the case of the complete sphere, negative powers of x must be rejected, so that the only non-vanishing coefficients in (38) are A_{i-2}, A_i, B_{i-2}, B_i, C_i, D_i; these being connected by four relations included among (39). There remain two arbitrary constants for satisfying the surface conditions (26) and (27).

§ 6. *Case in which the Potential is a Harmonic Function.*

If W is a harmonic function, each term $R_i S_i$ in its development must satisfy Laplace's equation, so that

$$R_i = Ar^i + Br^{-i-1}.$$

Rejecting the negative power, we assume R_i or R to have the form Ar^i. Equations (32) and (33) then reduce to

$$(40) \qquad (\lambda + 2\mu)y = -\frac{i}{(2i+1)x^{i+1}} \int \rho \frac{d(Rx^{i+1})}{dx} dx,$$

$$(41) \qquad \mu z = -\frac{1}{(2i+1)x^{i+1}} \int \rho \frac{d(Rx^{i+1})}{dx} dx.$$

The further discussion will be restricted to the case in which $i = 2$.

§ 7. *Sphere Strained by Centrifugal Force.*

If the bodily forces are the centrifugal forces due to uniform rotation about a fixed axis, we have

$$(42) \qquad W = -\frac{cg}{3a} r^2 P_2,$$

in which c denotes the ratio of equatorial centrifugal force to gravity, and P_2 is the zonal harmonic $\frac{3}{2} \cos^2 \theta - \frac{1}{2}$. We now take

$$(43) \qquad S_2 = -\frac{2}{3}P_2 = \frac{1}{3} - \cos^2 \theta,$$

$$(44) \qquad R_2 = R = \frac{cgr^2}{2a} = \frac{cga}{2} x^2.$$

In accordance with the notation of a former paper,[*] let e denote the ellipticity of a surface originally spherical and of radius r, and α the angular displacement of a radius vector for which $\theta = 45°$. With the above value of S_2, we have

$$(45) \qquad e = \frac{u}{r}, \qquad \alpha = \frac{v}{r}.$$

In writing the complete solutions of the differential equations for this case, we take A_0 and A_2 as arbitrary constants, expressing B_0, B_2, C_2, D_2 in terms of them by means of (39). We thus obtain the equations

$$(46) \qquad y = -\frac{7\mu}{\lambda} A_2 x^2 - \frac{cga}{(\lambda + 2\mu)x^3} \int_0^x \rho x^4 dx,$$

$$(47) \qquad z = \frac{7(\lambda + 2\mu)}{3\lambda} A_2 x^2 - \frac{cga}{2\mu x^3} \int_0^x \rho x^4 \, dx,$$

$$(48) \qquad \begin{aligned} e = A_0 + A_2 x^2 + \frac{cga}{30\mu(\lambda + 2\mu)} &\left[\frac{5(3\lambda + 5\mu)}{x^3} \int_0^x \rho x^4 \, dx \right. \\ &\left. - \frac{9(\lambda + \mu)}{x^5} \int_0^x \rho x^6 \, dx - 2(3\lambda + 8\mu) \int_0^x \rho x \, dx \right], \end{aligned}$$

$$(49) \qquad \begin{aligned} \alpha = \tfrac{1}{2} A_0 + \frac{5\lambda + 7\mu}{6\lambda} A_2 x^2 \\ + \frac{cga}{30\mu(\lambda + 2\mu)} &\left[\frac{5\mu}{x^3} \int_0^x \rho x^4 \, dx + \frac{3(\lambda + \mu)}{x^5} \int_0^x \rho x^6 \, dx - (3\lambda + 8\mu) \int_0^x \rho x \, dx \right]. \end{aligned}$$

Substituting these values in (26) and 27), we obtain the following equations for determining A_0 and A_2:

$$(50) \qquad \begin{aligned} 2A_0 - A_2 + \frac{cga}{30\mu(\lambda + 2\mu)} &\left[-10(9\lambda + 10\mu) \int_0^1 \rho x^4 \, dx \right. \\ &\left. + 72(\lambda + \mu) \int_0^1 \rho x^6 \, dx - 4(3\lambda + 8\mu) \int_0^1 \rho x \, dx \right] = 0, \end{aligned}$$

$$(51) \qquad \begin{aligned} A_0 + \frac{8\lambda + 7\mu}{3\lambda} A_2 + \frac{cga}{30\mu(\lambda + 2\mu)} &\left[5(3\lambda + 2\mu) \int_0^1 \rho x^4 \, dx \right. \\ &\left. - 24(\lambda + \mu) \int_0^1 \rho x^6 \, dx - 2(3\lambda + 8\mu) \int_0^1 \rho x \, dx \right] = 0. \end{aligned}$$

Solving, we find

$$(52) \qquad \begin{aligned} A_0 = \frac{cga}{30\mu(\lambda + 2\mu)(19\lambda + 14\mu)} &\left[25(27\lambda^2 + 56\lambda\mu + 28\mu^2) \int_0^1 \rho x^4 \, dx \right. \\ &\left. - 504(\lambda + \mu)^2 \int_0^1 \rho x^6 \, dx + 2(3\lambda + 8\mu)(19\lambda + 14\mu) \int_0^1 \rho x \, dx \right], \end{aligned}$$

$$(53) \qquad A_2 = \frac{cga}{30\mu(\lambda + 2\mu)(19\lambda + 14\mu)} \cdot 360\lambda(\lambda + \mu) \left(\int_0^1 \rho x^6 \, dx - \int_0^1 \rho x^4 \, dx \right).$$

Of especial interest are the surface values of e and α:

$$(54) \qquad e_1 = \frac{cga}{2\mu(19\lambda + 14\mu)} \left[5(8\lambda + 7\mu) \int_0^1 \rho x^4 \, dx - 21(\lambda + \mu) \int_0^1 \rho x^6 \, dx \right],$$

$$(55) \qquad \alpha_1 = \frac{cga}{4\mu(19\lambda + 14\mu)} \left[5\lambda \int_0^1 \rho x^4 \, dx + 14(\lambda + \mu) \int_0^1 \rho x^6 \, dx \right].$$

If ρ is constant it is easy to show that the foregoing equations give the following well-known results:

$$(56) \qquad e = \frac{c}{2(19\lambda + 14\mu)} \left[2(4\lambda + 3\mu) - (3\lambda + 2\mu)x^2 \right] \frac{\rho g a}{\mu},$$

$$(57) \qquad \alpha = \frac{c}{4(19\lambda + 14\mu)} \left[2(4\lambda + 3\mu) - (5\lambda + 4\mu)x^2 \right] \frac{\rho g a}{\mu}.$$

§ 8. *Laplace's Law of Density.*

To illustrate the application of the above formulas to a case of variable density, assume Laplace's law, expressed by the formula

$$(58) \qquad \rho = \rho_0 \frac{\sin \kappa x}{\kappa x} = \frac{\rho_m \kappa^3}{3(\sin \kappa - \kappa \cos \kappa)} \cdot \frac{\sin \kappa x}{\kappa x},$$

in which ρ_m is the mean density and ρ_0 the value of ρ at the center. We then find

$$(59) \qquad \int_0^x \rho x \, dx = \frac{\rho_m \kappa}{3(\sin \kappa - \kappa \cos \kappa)} (1 - \cos \kappa x),$$

$$(60) \quad \frac{1}{x^3} \int_0^x \rho x^4 \, dx = \frac{\rho_m \kappa}{3(\sin \kappa - \kappa \cos \kappa)} \left[\frac{6 - \kappa^2 x^2}{\kappa^2 x^2} \cos \kappa x - \frac{6 - \kappa^2 x^2}{\kappa^3 x^3} \sin \kappa x \right],$$

$$(61) \quad \frac{1}{x^5} \int_0^x \rho x^6 \, dx = \frac{\rho_m \kappa}{3(\sin \kappa - \kappa \cos \kappa)} \left[\frac{120 - 60\kappa^2 x^2 + 5\kappa^4 x^4}{\kappa^5 x^5} \sin \kappa x \right.$$
$$\left. - \frac{120 - 20\kappa^2 x^2 + \kappa^4 x^4}{\kappa^4 x^4} \cos \kappa x \right].$$

If the ratio of mean density to surface density is assumed to be 2, we find $\kappa = 2.461$, and

$$(62) \qquad \int_0^1 \rho x^4 \, dx = .168 \rho_m, \qquad \int_0^1 \rho x^6 \, dx = .108 \rho_m.$$

These values substituted in (54) and (55) give

$$(63) \qquad \frac{e_1}{c} = \frac{4.45\lambda + 3.61\mu}{2(19\lambda + 14\mu)} \cdot \frac{\rho_m g a}{\mu},$$

$$(64) \qquad \frac{\alpha_1}{c} = \frac{2.35\lambda + 1.51\mu}{4(19\lambda + 14\mu)} \cdot \frac{\rho_m g a}{\mu}.$$

Comparing with the surface values of e and α for a sphere of uniform density ρ_m,

$$(65) \qquad \frac{e_1}{c} = \frac{5\lambda + 4\mu}{2(19\lambda + 14\mu)} \cdot \frac{\rho_m g a}{\mu},$$

$$(66) \qquad \frac{\alpha_1}{c} = \frac{3\lambda + 2\mu}{4(19\lambda + 14\mu)} \cdot \frac{\rho_m g a}{\mu},$$

it appears that the assumption of variable density decreases e_1 by about 11 per cent and α_1 by about 22 per cent.

We may also compare the values at the center, which with Laplace's law of density are found to be

$$(67) \qquad \frac{2\alpha_0}{c} = \frac{e_0}{c} = \frac{A_0}{c} = \frac{124.4\lambda^2 + 349.1\lambda\mu + 191.8\mu^2}{30(\lambda + 2\mu)(19\lambda + 14\mu)} \cdot \frac{\rho_m g a}{\mu},$$

while the corresponding values for uniform density ρ_m are

$$(68) \qquad \frac{2\alpha_0}{c} = \frac{e_0}{c} = \frac{4\lambda + 3\mu}{19\lambda + 14\mu} \frac{\rho_m g a}{\mu} = \frac{120\lambda^2 + 330\lambda\mu + 180\mu^2}{30(\lambda + 2\mu)(19\lambda + 14\mu)} \cdot \frac{\rho_m g a}{\mu}.$$

The central values of e and α are thus slightly increased by the assumption of variable density.

§ 9. Gravitating Sphere.

The results of the above solution cannot be applied directly to a gravitating body of size and mass comparable with those of the earth, since the changes in the gravitational forces caused by the strain will be of the same order of importance as the disturbing forces. There is no difficulty in expressing the gravitational forces in terms of the strain and the density and thus forming the differential equations which must replace (16) and (17) above given, but the solution of these equations appears to involve serious analytical difficulties.*

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* In expressing the equations of interior equilibrium in terms of the strain, for the case of a gravitating body, it is necessary to assume that the changes of stress and strain from their initial values are related in the same way as if the initial condition were one of zero stress and strain. This assumption is the basis of the solution given for the case of a compressible sphere in the paper cited above, the reasoning being in this respect substantially identical with that previously applied to the case of incompressibility. (See A. E. H. Love, A Treatise on the Mathematical Theory of Elasticity, 2d ed., p. 249.) It has sometimes been assumed that this reasoning is valid only in the case of incompressibility. Thus Love states that if the sphere is homogeneous and incompressible, the initial stress may be taken to be of the nature of a hydrostatic pressure, its value being then easily determined, and the terms which correspond to this initial stress in the equations of equilibrium being found to cancel with those representing the initial body-force. It seems clear, however, first that even in an incompressible body the initial condition may involve tangential stress, and secondly that the assumption of no initial tangential stress is not essential to the argument. Whatever values the initial stress-components may have, the terms representing them in the equations of equilibrium will cancel the terms representing the initial body-forces.

If the body is assumed incompressible and of uniform density, the effect of gravitation can be taken account of synthetically by a method employed by Kelvin.*   This method consists in combining the results for the two cases of a gravitating fluid and a non-gravitating elastic solid.   It does not appear that similar reasoning can be applied if either of the assumptions of incompressibility and uniform density is dropped, since the distribution of the forces of gravitation cannot be assumed to be the same for the strained elastic solid as for the fluid.

The method of applying this principle in the solutions cited above, both for the case of incompressibility and for that of compressibility, consists in comparing the forces acting upon a volume-element in the strained condition with those initially acting upon the same volume-element.   It would perhaps be more reasonable to deal with an individual element of matter, using the increments of the stress-components acting on that element in expressing the stress-strain relations by the usual formulas.   The increment of stress for an individual element would be in equilibrium with the increment of body-force acting upon that element, which would consist in part of the change in the gravitational force due to the displacement of the attracted element. This would introduce additional terms into the equations of equilibrium, but would add no difficulty to their solution.

* THOMSON and TAIT's *Natural Philosophy*, § 840, Ed. of 1890.